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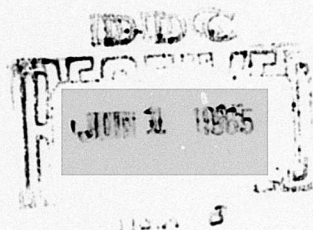
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## On Newton's Method



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ON NEWTON'S METHOD

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## ABSTRACT

In this note we discuss Newton's method in a setting somewhat more restrictive than customary. In this setting, however, we claim to have proved superlinear convergence of the Newton process without assuming twice differentiability or Lipschitz continuity of the first derivative of the operator. A further feature is that the iteration to be discussed is not initially but is eventually the Newton process. With this feature global rather than local convergence is achieved.

The literature on Newton's method after the fundamental paper of Kantorovich [1] is large. Pertinent to this note is the work of Schröder [2], in which the first derivative is assumed to be Lipschitz continuous and quadratic convergence ensues. Another related paper showing the effectiveness of Newton's method with once differentiability may be found in the literature, but this claim was later retracted by the author. In Schröder [2], a long list of references to Newton's method may be found.

Let  $Q$  be a differentiable non-linear operator with domain and range in a Hilbert space  $H$ . Given  $x_0$  arbitrarily in  $H$ , set  $f(x) = \int_0^1 [Q(x_0 + t(x - x_0)), x - x_0] dt$ , and  $S = \{x \in H: f(x) < 0\}$ . Let  $f'(x, h)$  denote the  $F$ -differential ( $F = \text{Fréchet}$ ) of  $f$ . Take  $\nabla f(x)$  in  $H$  so that  $[\nabla f(x), h] = f'(x, h)$  for all  $h \in H$ . Let  $Q'(x, h)$  denote the  $F$ -differential of  $Q$ . Let  $Q'(x, h) = J(x)h$ . Thus for each  $x$ ,  $J(x)$  is a linear operator on  $H$ .

Lemma. [3] Assume  $J(\cdot)$  is continuous on  $S$  and  $J(x)$  is self-adjoint for all  $x \in S$ . Then  $Q(x) = \nabla f(x)$ .

Proof. We shall prove that for  $x \in S$  and  $h \in H$ ,  
 $f'(x, h) = [Q(x), h]$ . By calculation  $f'(x, h) =$   
 $= \int_0^1 \{ [Q'(x_0 + t(x - x_0), th), x - x_0] + [Q(x_0) + t(x - x_0), h] \} dt,$

and  $[Q(x), h] = \int_0^1 \frac{d}{dt} [Q(x_0 + t(x - x_0)), th] dt =$   
 $= \int_0^1 \{ [Q'(x_0 + t(x - x_0)), x - x_0], th \} + [Q(x_0 + t(x - x_0)), h] \} dt =$   
 $f'(x, h)$ . Q.E.D.

For the next theorem we set  $\Delta(x, \rho) = f(x) - f(x - \rho J^{-1}(x)Q(x))$   
 and  $g(x, \rho) = \Delta(x, \rho) / \rho [Q(x)J^{-1}(x)Q(x)]$ . We shall abbreviate  
 $Q(x_k)$ ,  $J(x_k)$ , etc., by  $Q_k$  and  $J_k$ , respectively.

Theorem. Assume  $J(\cdot)$  continuous on  $S$ . For each  $x \in S$   
 assume  $J(x)$  self-adjoint and  $\mu \|h\|^2 \leq [h, J(x)h] \leq \lambda \|h\|^2$  for  
 all  $h \in H$  and some  $\mu > 0$ . Set  $x_{k+1} = x_k - \rho_k J_k^{-1} Q_k$ . Here  
 $\rho_k$  is chosen so that for  $\delta < \frac{1}{2}$ ,  $0 < \delta \leq g(x_k, \rho_k) \leq 1 - \delta$ , with  
 $\rho_k = 1$ , if possible. Then:

(a) For arbitrary  $x^0$  the sequence  $\{x^k\}$  converges to the  
 unique root of  $Q$ .

(b) There exists a number  $N$  such that if  $k > N$  then  
 $\rho_k = 1$ .

(c) The convergence of  $\{x_k\}$  is superlinear--that is to  
 say it is faster than any geometric progression.

Proof. We show first that if  $Q(x_k) \neq 0$  and  $x_k \in S$   
 then  $\rho_k$  can be chosen as claimed. By Taylor's theorem  
 $\Delta(x, \rho) = f(x) - f(x - \rho J^{-1}(x)Q(x)) = \rho [Q(x), J^{-1}(x)Q(x)] -$   
 $-\rho^2 [J^{-1}(x)Q(x), J(\xi(\rho))J^{-1}(x)Q(x)]/2$  where  $\xi(\rho)$  lies "between"  
 $x$  and  $x - \rho J^{-1}(x)Q(x)$ . Hence  $g(x, \rho) =$   
 $1 - \rho [J^{-1}(x)Q(x)J(\xi(\rho))J^{-1}(x)Q(x)]/2[Q(x)J^{-1}(x)Q(x)]$ . It follows  
 that  $g(x, \cdot)$  is continuous at each  $\rho$  for which  $\xi(\rho) \in S$ . In  
 particular  $g(x, 0) = 1$ . If  $S$  were bounded, then we observe

that there would exist a least positive number, say  $\delta$ , for which  $g(x, \delta) = 0$ , with  $\rho < \delta$  implying  $f(\rho) \in S$ . Hence if  $S$  is bounded, there exists  $\rho > 0$  such that  $\delta \leq g(x, \rho) \leq 1 - \delta$ .

Suppose then  $S$  were unbounded. If so, there exists a sequence

$\{x_k\} \in S$  such that  $\{\|x_k\|\} \rightarrow \infty$ , while  $f(x_k) < 0$  for all  $k$ .

By Taylor's theorem if  $u \in S$ ,  $f(x_k) \geq f(u) + \|x_k - u\|[\|x_k - u\|/2] - \|\nabla f(u)\|$  showing that  $f(x_k) \geq 0$  for large  $k$ . Thus the numbers  $\rho_k$  are well defined.

Observe next that  $\Delta(x, \rho) \geq [Q(x)J^{-1}(x)Q(x)][1 - \rho\lambda^2/\mu^2]$

for all  $\rho$  such that the line segment  $[x, x - \rho J^{-1}(x)Q(x)]$

belongs to  $S$ . To prove this we apply the inequalities

$$\mu\|h\|^2 \leq [h, J(x)h] \leq \lambda\|h\|^2 \quad \text{and} \quad \|h\|^2/\lambda \leq [h, J^{-1}(x)h] \leq \|h\|^2/\mu$$

to the above Taylor expansion of  $\Delta(x, \rho)$ . It follows that

$g(x_k, \rho_k) \geq [1 - \rho_k\lambda^2/2\mu^2]$  and therefore  $\rho_k \geq 2\delta\mu^2/\lambda^2$ . Thus

$\Delta(x_k, \rho_k) \geq 2\|Q_k\|^2\delta\mu^2/\lambda^3$ . Suppose now that  $\{\|Q_k\|\} \not\rightarrow 0$ . Then a subsequence of  $\{\Delta(x_k, \rho_k)\} \rightarrow -\infty$ , and thus  $\{f(x_k)\} \rightarrow -\infty$ .

Observe that  $f$  is continuous on  $\bar{S}$  and that since  $f$  is convex, the set  $S_m = \{x \in \bar{S}: f(x) \leq m\}$  is closed and convex for each  $m$ .

Thus the set  $S_m$  is weakly closed and the function  $f$  is weakly lower semicontinuous. Since  $\bar{S}$  is weakly compact,  $f$  achieves a minimum on  $\bar{S}$ , which contradicts that  $\{f(x_k)\} \rightarrow -\infty$ . It follows

therefore that  $\|Q_k\| \rightarrow 0$ , and  $f(x_k) \rightarrow$  to a limit  $L$ . Next we

prove that  $\{x_k\}$  is Cauchy. By Taylor's theorem, if  $s > k$ ,

$$f(x^s) - f(x^k) \geq [Q_k, x^s - x^k] + \mu\|x^s - x^k\|^2/2.$$

Since  $S$  is bounded  $\|x^s - x^k\| \leq B$  where  $B$  is the diameter of  $S$ . Thus

$\|x^s - x^k\|^2 \leq (f(x^s) - f(x^k) + \|Q_k\|B)2/\mu$ . Since  $\{f(x^k)\}$  is Cauchy and  $\|Q_k\| \rightarrow 0$ ,  $\{x^s\}$  is Cauchy. Thus  $\{x^k\} \rightarrow z$  and  $Q(z) = 0$ . Suppose now that  $z$  is not unique, so that  $Q(z_1) = Q(z_2) = 0$ . Thus  $f(z_1) - f(z_2) \geq \mu\|z_1 - z_2\|^2$ ,  $f(z_2) - f(z_1) \geq \mu\|z_2 - z_1\|^2$ , and  $\mu\|z_2 - z_1\|^2 \leq f(z_2) - f(z_1) \leq -\mu\|z_2 - z_1\|^2$ , which is a contradiction unless  $z_1 = z_2$ .

To prove (b), we observe that by setting  $J(\xi(\rho)) = J(x) + J(\xi(\rho)) - J(x)$  that  $g(x, \rho) = 1 - \rho/2 - \rho[J^{-1}(x)Q(x), (J(\xi(\rho)) - J(x))J^{-1}(x)Q(x)]/2[Q(x), J^{-1}(x)Q(x)]$ . Thus  $|g(x, \rho) - 1 + \rho/2| \leq \lambda\rho\|J(\xi(\rho)) - J(x)\|/2\mu^2$ . Since  $\{x_k\}$  is Cauchy,  $\{\|x_{k+1} - x_k\|\}$  and  $\{\|\xi(\rho_k) - x_k\|\}$  converge to 0. Indeed if  $\xi(\rho_k) = \xi_k$  then  $\{x_0, \xi_0, x_1, \xi_1, \dots\}$  is also a Cauchy sequence; and it, together with  $\{z\}$  is a compactum. Consequently on this compactum  $J$  is uniformly continuous. Therefore  $\|J(\xi_k) - J(x_k)\| \rightarrow 0$ . Let  $\varepsilon_k = \lambda\|J(\xi_k) - J(x_k)\|/2\mu^2$  and assume that  $k$  is taken sufficiently large so that  $\varepsilon_k < 1/2$ . Then  $|g(x_k, \rho_k) - 1 + \rho_k/2| \leq \varepsilon_k \rho_k$  implies

$$\frac{1 - g(x_k, \rho_k)}{1/2 + \varepsilon_k} \leq \rho_k \leq \frac{1 - g(x_k, \rho_k)}{1/2 - \varepsilon_k}.$$

Thus eventually  $\rho_k$  can be any point which satisfies  $2\delta < \rho_k \leq 2(1 - \delta)$ . Therefore since  $2\delta < 1$ , the choice  $\rho_k = 1$  will be eventually feasible.

To prove (c) we write  $x_{k+1} - z = x_k - z - \rho_k J^{-1}(x_k)Q(x_k) = x_k - z - \rho_k J^{-1}(x_k)J(x_k)(x_k - z) + \rho_k J^{-1}(x_k)[J(x_k)(x_k - z) - Q(x_k)]$ . By hypothesis  $Q$  is Fréchet differentiable at  $x_k$ , consequently

given  $\epsilon > 0$  there exists  $\delta > 0$  such that  $||Q(z) - Q(x_k) - J(x_k)(z - x_k)|| < \epsilon ||z - x_k||$ , whenever  $||z - x_k|| < \delta$ .

Thus we choose  $k$  so large that this latter inequality is

fulfilled we get  $||x_{k+1} - z|| \leq ||x_k - z|| (1 - \rho_k + \rho_k \epsilon/\mu)$ .

If furthermore  $k \geq N$ ,  $||x_{k+1} - z|| \leq \epsilon ||x_k - z||/\mu$ , showing the superlinear convergence.

## REFERENCES

- [1] Kantorovich, L.V., Functional Analysis and Applied Mathematics, Uspekhi Matematicheskikh Nauk, Vol. III, No. 6 pp. 89-185, 1948. Also NBS Report 1509, Edited by G. E. Forsythe.
- [2] Schröder, J. Über das Newtonsche Verfahren, Archive for Rational Mechanics and Analysis, 1957, p. 154-180.
- [3] M. Kerner, Die Differentiale in der allgemeinen Analysis, Ann. of Math. V. 34, 1933, p. 546-572. See also [4].
- [4] Vainberg, M. M., Variational Methods for the Study of Nonlinear Operators, Holden Day, San Francisco, 1964.

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